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# Exact relations for two-component Fermi gases with contact interactions

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# Outline

- The system
- Hamiltonian
- Energy functional
- The contact
- Other exact relations
- Other approaches & results
- Experimental tests
- Other systems

# The system

$N$  fermionic atoms populating two spin states  
(up and down, or blue dots and red dots).

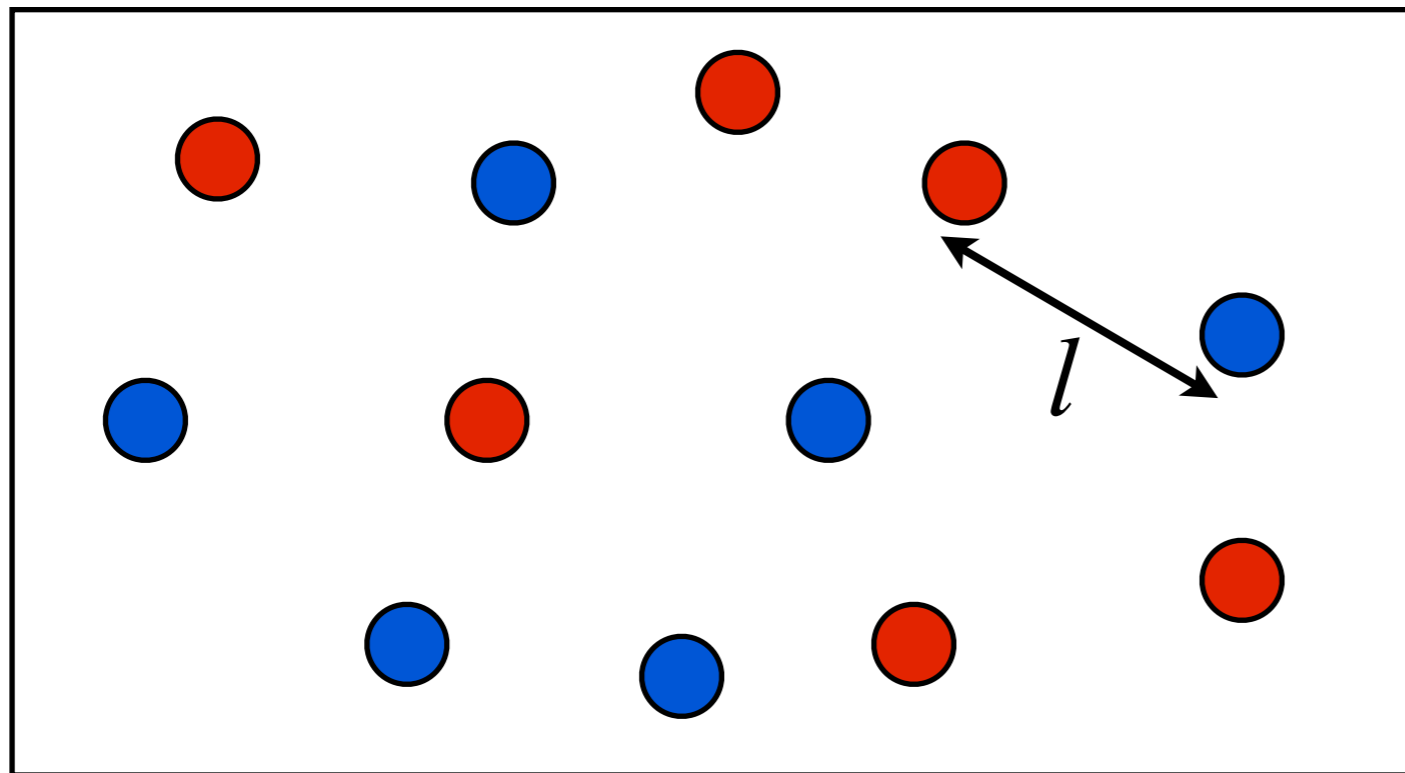
( $N$  is arbitrary)

$r_{vdw} \sim$  a few nanometers

Ultradilute:  $l \gg r_{vdw}$

Ultracold:  $\lambda_{dB} \gg r_{vdw}$

Strongly interacting:  $|a| \gg r_{vdw}$



Can take the  
idealized model:

*contact interaction*

$(r_{vdw} = 0)$

# The Hamiltonian

$$H = \sum_{\mathbf{k}\sigma} \frac{\hbar^2 k^2}{2m} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + \frac{4\pi\hbar^2 a}{m\Omega} \sum_{\mathbf{q}\mathbf{k}\mathbf{k}'} \Lambda(\mathbf{k}') c_{\mathbf{q}/2+\mathbf{k}\uparrow}^\dagger c_{\mathbf{q}/2-\mathbf{k}\downarrow}^\dagger c_{\mathbf{q}/2-\mathbf{k}'\downarrow} c_{\mathbf{q}/2+\mathbf{k}'\uparrow}$$

$$+ \int d^3r \sum_{\sigma} V(\mathbf{r}) \psi_{\sigma}^\dagger(\mathbf{r}) \psi_{\sigma}(\mathbf{r}).$$

*(also good model for dilute neutron matter)*

$a$  : scattering length

$\Omega$  : large volume

$V(\mathbf{r})$  : external potential

$$\Lambda(\mathbf{k}) = 1 \text{ for all finite } \mathbf{k}, \text{ BUT } \int d^3k \frac{\Lambda(\mathbf{k})}{k^2} \equiv 0,$$

a spinoff of the Huang-Yang pseudopotential,

$$U(r) = \frac{4\pi\hbar^2 a}{m} \delta(\mathbf{r}) \frac{\partial}{\partial r} r$$

# The pseudopotential

$$H_{\text{int}} = \frac{4\pi\hbar^2 a}{m} \int d^3 r_1 d^3 r_2 \psi_{\uparrow}^{\dagger}(\mathbf{r}_1) \psi_{\downarrow}^{\dagger}(\mathbf{r}_2) \delta(\mathbf{r}_1 - \mathbf{r}_2) \frac{\partial}{\partial r_{12}} [r_{12} \psi_{\downarrow}(\mathbf{r}_2) \psi_{\uparrow}(\mathbf{r}_1)]$$

mimicing Eq. (38) of *K. Huang and C. N. Yang, Phys. Rev. 105, 767 (1957)*

$\delta(\mathbf{r}) \frac{\partial}{\partial r} r$  behaves almost like the delta function, except  $\delta(\mathbf{r}) \frac{\partial}{\partial r} r \frac{1}{r} = 0$

So I introduced a generalized function  $\lambda(\mathbf{r})$  satisfying

$$\lambda(\mathbf{r}) = 0, \quad \mathbf{r} \neq 0. \quad \int d^3 r \lambda(\mathbf{r}) = 1, \quad \int d^3 r \lambda(\mathbf{r}) \frac{1}{r} = 0$$

Its Fourier transform  $\Lambda(\mathbf{k}) \equiv \int d^3 r \lambda(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}}$  satisfies

$$\Lambda(\mathbf{k}) = 1, \quad k < \infty; \quad \text{but} \quad \int \frac{d^3 k}{(2\pi)^3} \Lambda(\mathbf{k}) \frac{4\pi}{k^2} = 0$$

$$\text{SO } H_{\text{int}} = \frac{4\pi\hbar^2 a}{m\Omega} \sum_{\mathbf{q}\mathbf{k}\mathbf{k}'} c_{\mathbf{q}/2+\mathbf{k}\uparrow}^{\dagger} c_{\mathbf{q}/2-\mathbf{k}\downarrow}^{\dagger} c_{\mathbf{q}/2-\mathbf{k}'\downarrow} c_{\mathbf{q}/2+\mathbf{k}'\uparrow} \Lambda(\mathbf{k}')$$

*Tan, Ann. of Phys. 323 (2008), 2952*

$$H = \sum_{\mathbf{k}\sigma} \frac{\hbar^2 k^2}{2m} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + \frac{4\pi\hbar^2 a}{m\Omega} \sum_{\mathbf{q}\mathbf{k}\mathbf{k}'} \Lambda(\mathbf{k}') c_{\mathbf{q}/2+\mathbf{k}\uparrow}^\dagger c_{\mathbf{q}/2-\mathbf{k}\downarrow}^\dagger c_{\mathbf{q}/2-\mathbf{k}'\downarrow} c_{\mathbf{q}/2+\mathbf{k}'\uparrow} \\ + \int d^3r \sum_{\sigma} V(\mathbf{r}) \psi_{\sigma}^\dagger(\mathbf{r}) \psi_{\sigma}(\mathbf{r})$$

implies the *Bethe-Peierls boundary condition*:

$$\{N\text{-body wave function}\} \propto \frac{1}{r} - \frac{1}{a} + O(r), \quad r \rightarrow 0$$

$r$  : distance between a spin-up fermion & a spin-down fermion

This  $1/r$  singularity implies that  $n_{\mathbf{k}\sigma} \rightarrow \frac{C}{k^4}, \quad k \rightarrow \infty$

$$\sigma = \uparrow, \downarrow$$

# Energy functional

Taking the expectation value of  $H$ ,  
and using the Bethe-Peierls boundary condition, we get

$$E = \frac{\hbar^2 \Omega C}{4\pi a m} + \int \frac{\Omega d^3 k}{(2\pi)^3} \sum_{\sigma} \frac{\hbar^2 k^2}{2m} \left( n_{\mathbf{k}\sigma} - \frac{C}{k^4} \right) + \int d^3 r \sum_{\sigma} V(\mathbf{r}) \rho_{\sigma}(\mathbf{r})$$

$$n_{\mathbf{k}\sigma} \equiv \langle c_{\mathbf{k}\sigma}^{\dagger} c_{\mathbf{k}\sigma} \rangle \quad C \equiv \lim_{k \rightarrow \infty} k^4 n_{\mathbf{k}\sigma}$$

Energy is a **linear** functional of momentum distribution  $n_{\mathbf{k}\sigma}$   
(and density distribution in the presence of external potential)

*Tan, Ann. of Phys. 323 (2008), 2952*

# Cancellation of Ultraviolet Divergences

$$E = \frac{\hbar^2 \Omega C}{4\pi a m} + \int \frac{\Omega d^3 k}{(2\pi)^3} \sum_{\sigma} \frac{\hbar^2 k^2}{2m} \left( n_{\mathbf{k}\sigma} - \frac{C}{k^4} \right) + \int d^3 r \sum_{\sigma} V(\mathbf{r}) \rho_{\sigma}(\mathbf{r})$$

kinetic energy  $\int \frac{\Omega d^3 k}{(2\pi)^3} \sum_{\sigma} \frac{\hbar^2 k^2}{2m} n_{\mathbf{k}\sigma} \rightarrow +\infty$

interaction energy  $\rightarrow -\infty$

but

$$\{\text{kinetic energy}\} + \{\text{interaction energy}\} = \text{finite}$$



# The contact

$$E = \frac{\hbar^2 \Omega C}{4\pi a m} + \int \frac{\Omega d^3 k}{(2\pi)^3} \sum_{\sigma} \frac{\hbar^2 k^2}{2m} \left( n_{\mathbf{k}\sigma} - \frac{C}{k^4} \right) + \int d^3 r \sum_{\sigma} V(\mathbf{r}) \rho_{\sigma}(\mathbf{r})$$

$$n_{\mathbf{k}\sigma} \rightarrow \frac{C}{k^4}, \quad k \rightarrow \infty \quad \text{due to the contact interaction}$$

The contact:  $\mathcal{I} = \Omega C = \int C(\mathbf{r}) d^3 r$

$C(\mathbf{r})$  : local contact density

# Physical meaning of the contact

density-density correlation function

$$\langle \rho_{\uparrow}(\mathbf{r}_c - \mathbf{r}/2) \rho_{\downarrow}(\mathbf{r}_c + \mathbf{r}/2) \rangle = \frac{C(\mathbf{r}_c)}{16\pi^2} \frac{1}{r^2} + O(1/r), \quad r \rightarrow 0$$

$$\text{Contact } \mathcal{I} = \int C(\mathbf{r}_c) d^3 r_c$$

So the contact is, roughly speaking,  
the chance of finding two atoms close to each other

# A simplified energy functional

$\epsilon_\nu$  : single-particle energy levels in the external potential

$n_\nu$  : population of the  $\nu$ th level

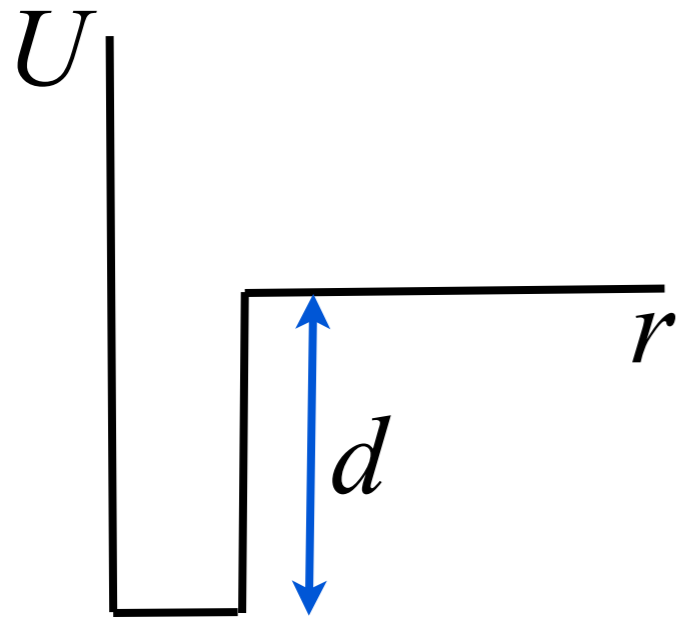
$$E = E[n_\nu] = \frac{\hbar^2 \mathcal{I}}{4\pi a m} + \lim_{\epsilon_{\max} \rightarrow \infty} \left( \sum_{\epsilon_\nu < \epsilon_{\max}} \epsilon_\nu n_\nu - \frac{\hbar \mathcal{I}}{\pi^2} \sqrt{\frac{\epsilon_{\max}}{2m}} \right)$$

valid in free space, harmonic traps, optical lattices, etc.

*Tan, Phys. Rev. Lett. 107, 145302 (2011)*

Unlike the **Density Functional** in electronic structure, the above functional is *explicitly known*, and is *valid for all quantum states* (not just the  $N$ -body ground state).

# What if we tune the interaction in real time?



Can fine-tune depth  $d$  to produce large  $a$ .

If  $a$  is large,  $\Delta(1/a) \propto \Delta d$

{change of energy} = {change of interaction}  $\times$   
{probability inside the range of interaction}

$$\frac{dE}{dt} = \frac{\hbar^2 \mathcal{I}(t)}{4\pi m} \frac{d(-1/a)}{dt}$$

*dynamic sweep relation*

*Tan, Ann. of Phys. 323 (2008), 2971*

# Pressure relation (in free space)

$$\text{Pressure} = \frac{2}{3} \{\text{energy density}\} + \frac{\hbar^2 C}{12\pi am}$$

*Tan, Ann. of Phys. 323 (2008), 2987*

# Generalized virial theorem (harmonic trap)

$$E = 2E_V - \frac{\hbar^2 \mathcal{I}}{8\pi am}$$

which generalizes the virial theorem at unitarity  
as found by John Thomas

*Tan, Ann. of Phys. 323 (2008), 2987*

# Other theoretical approaches and results

- Operator product expansion (**Eric Braaten & D. Kang & L. Platter et al**); two-body decay rate (ST, E. Braaten & H.-W. Hammer et al)
- $N$ -body wave functions (**F. Werner & Y. Castin & L. Tarruell, R. Combescot & F. Alzetto & X. Leyronas**)
- Thermodynamics, RF-Spectroscopy and closed-channel molecules (**S. Zhang & A.J. Leggett**)
- Theoretical analyses of the contact (H. Hu & X.J. Liu et al)
- Large- $N$  expansion (T. Enss)
- few-body numerical tests (D. Blume et al)
- Quantum Monte-Carlo (J. E. Drut et al, G. Bertaina & S. Giorgini, S. Gandolfi & K.E. Schmidt & J. Carlson, K. Van Houcke & F. Werner & E. Kozik & N. Prokof'ev & B. Svistunov)
- Critical behaviors of contact (Y.Y. Chen & Y.Z. Jiang & X.W. Guan & Q. Zhou)

# Other relations involving the contact

- Clock shift (M. Punk & W. Zwerger, G. Baym & C.J. Pethick & Z. Yu & M.W. Zwerlein, C. Langmack & M. Barth & W. Zwerger & E. Braaten)
- Tail of radio-frequency spectrum at large detunings (W. Schneider & V. B. Shenoy & M. Randeria, M. Punk & P.T. Dumitrescu & W. Zwerger, E. Braaten & D. Kang & L. Platter, P. Pieri & A. Perali & G.C. Strinati)
- Static structure factor at large  $q$   
(Hui Hu & Xiaji Liu & P. Drummond et al)
- Dynamic structure factor at large  $q$  or large  $\omega$  (D. T. Son & E. G. Thompson, E. Taylor & M. Randeria)
- Frequency dependent viscosity (E. Taylor & M. Randeria, T. Enss & R. Haussmann & W. Zwerger, W.D. Goldberger & Z.U. Khandker)
- Extension to arbitrary partial waves (M. He & S. Zhang & H.M. Chan & Q. Zhou)
- ...

# Experimental verifications & measurements

- **D. S. Jin group at JILA:**  
tail of momentum distribution, adiabatic relation,  
generalized virial theorem, tail of RF spectrum;  
measured contact as a function of temperature at unitarity
- **C. J. Vale group at Swinburne U. of Technology:**  
Bragg spectroscopy, static structure factor
- **R. Hulet group at Rice U.** (data analyzed by F. Werner &  
L. Tarruell & Y. Castin):  
photoassociation, # of closed channel molecules
- **C. Salomon group at ENS:** contact near unitarity
- ...



# Review article

Eric Braaten, *Universal Relations for Fermions with Large Scattering Length*, arXiv:1008.2922

# Other systems

- 2D gases
- 1D gases (M. Barth & W. Zwerger, X. Guan)
- Bose gases: 2-body contact and **3-body contact** (E. Braaten et al, Y. Castin & F. Werner, X.J. Liu & B. Mulkerin & L. He & H. Hu)
- Bose-Fermi mixture (Z.Q. Yu & S. Zhang & H. Zhai)
- 1-component Fermi gas with p-wave resonant interaction => **p-wave contact**  
(Z. Yu & J. Thywissen & S. Zhang, C. Luciuk & S. Trotzky & S. Smale & Z. Yu & S. Zhang & J.H. Thywissen, S.M. Yoshida & M. Ueda)

# Summary

- Dilute ultracold gases can often be approximated by *contact-interaction models*  
=> contact  
& lots of exact relations involving the contact